Time Consistent Multi-period Robust Risk Measures and Portfolio Selection Models with Regime-switching

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(Joint work with Jia Liu and Yongchang Hui)

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Introduction
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- Multi-period worst-case risk measure
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- Multi-period worst-case risk measure
- Regime dependent multi-period robust risk measures

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- Application to portfolio selection problems
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- Empirical illustrations
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Introduction

Traditional risk measure
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CVaR(x) = \inf_{\nu} \{ \nu + \epsilon^{-1} \mathbb{E}_P[ x - \nu ]_+ \},
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\( \epsilon \in (0, 1] \) is a given loss tolerant probability (say, 5%)
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★ The computation of risk measure relies on the underlying distribution $P$
Traditional distribution assumptions, such as normal or student’s t, does not fit the financial data well
Introduction (Cont’d)

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Sample average approximation (Shapiro et al. [2009])
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**Definition**

For given risk measure $\rho$, the worst-case risk measure with respect to $\mathcal{P}$ is defined as $w\rho(x) \triangleq \sup_{P \in \mathcal{P}} \rho(x)$.
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By constructing different uncertainty sets $\mathcal{P}$, we can derive different versions of worst-case risk measures.
Application of worst-case risk measures
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★ Above studies are all in static case
Introduction (Cont’d)

Multi-period robust optimization
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- Tractability, time consistency
Proper: dynamic information process $\rightarrow$ regime switching technique framework
Introduction (Cont’d)

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Our contributions
Introduction (Cont’d)

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- Discuss the time consistency of the new measures
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Proper: dynamic information process $\rightarrow$ regime switching technique framework

Our contributions

- Propose a new form of multi-period robust risk measure
- Propose two kinds of regime-based robust risk measure
- Discuss the time consistency of the new measures
- Apply to multi-stage portfolio selection problems and derive their analytical optimal solution or find tractable transformation
Basic setting
Multi-period worst-case risk measure

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- There are $T + 1$ time points and $T$ periods
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- Random loss process $\{x_t, \ t = 0, 1, \cdots, T\}$ is defined on the probability space $(\Omega, \mathcal{F}, P)$, and adapted to the filtration $\mathcal{F}_t, \ t = 0, 1, \cdots, T$
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- $\mathcal{L}_{t,T} = \mathcal{L}_t \times \cdots \times \mathcal{L}_T$.
- $x_{t,T} = (x_t, \cdots, x_T) \in \mathcal{L}_{t,T}$.
Typical multi-period risk measure

Considering the distributional uncertainty
\[\text{At each period } t, P_t \text{ is required to belong to an uncertainty set } P_t \text{ which contains all possible probability distributions of random loss } x_t \text{ and is observable at time point } t-1.\]

\[\text{\star P_1, P_2, \ldots, P_T are mutually independent.}\]
Typical multi-period risk measure

- A conditional mapping \( \rho_{t,T}(\cdot) : L_{t+1,T} \rightarrow L_t \)
Multi-period worst-case risk measure (Cont’d)

Typical multi-period risk measure

- A conditional mapping $\rho_{t,T}(\cdot) : \mathcal{L}_{t+1,T} \rightarrow \mathcal{L}_t$
- Separable expected conditional (SEC) mapping:

$$
\rho_{t,T}(x_{t+1,T}) = \sum_{i=t+1}^{T} \mathbb{E}_P \left[ \rho_{i,F_{i-1}}(x_i) | F_t \right], \quad t = 0, 1, \cdots, T - 1.
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Considering the distributional uncertainty

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At each period $t$, $P_t$ is required to belong to an uncertainty set $\mathcal{P}_t$ which contains all possible probability distributions of random loss $x_t$ and is observable at time point $t - 1$.

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Considering the distributional uncertainty

- At each period $t$, $\mathcal{P}_t$ is required to belong to an uncertainty set $\mathcal{P}_t$ which contains all possible probability distributions of random loss $x_t$ and is observable at time point $t - 1$.
- $\mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_T$ are mutually independent.
We obtain a robust estimation of the one-period conditional risk at period $t$: \( \sup_{P_t \in \mathcal{P}_t} \rho_{t|F_{t-1}} (x_t) \)
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Then all the estimations of risks at different periods are added together with respect to their conditional expectations.
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Multi-period worst-case risk measure (Cont’d)

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- Then all the estimations of risks at different periods are added together with respect to their conditional expectations

$\Rightarrow$ This gives us the multi-period worst-case risk measure.
Worst case risk measure

For $t = 0, 1, \cdots, T - 1$ and $x_{t+1,T} \in \mathcal{L}_{t+1,T}$,

$$w_{\rho_t,T}(x_{t+1,T}) = \sum_{i=t+1}^{T} \mathbb{E}_{P_i} \left[ \sup_{P_i \in \mathcal{P}_i} \rho_i |F_{i-1}(x_i) \right] | \mathcal{F}_t$$

is called the conditional worst-case risk mapping. The sequence of the risk mappings $\{w_{\rho_t,T}\}_{t=0}^{T-1}$ is called the multi-period worst-case risk measure.
Multi-period worst-case risk measure (Cont’d)

Dynamic formulation
Dynamic formulation

\[ w\rho_{t, T}(x_{t, T}) = \left( \sup_{P_t \in \mathcal{P}_t} \rho_t |\mathcal{F}_{t-1}(x_t) \right) + \mathbb{E}_{P_{t-1}} \left[ w\rho_{t, T}(x_{t+1, T}) |\mathcal{F}_{t-1} \right], \quad t = 1, 2, \cdots, T. \]
Multi-period worst-case risk measure (Cont’d)

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Compared with the adjustable robust optimization (ARO)

Compared with the adjustable robust optimization (ARO): makes worst-case estimation for both two parts. ⇒ The worst-case estimation will not be cumulated to the earlier period. Not that conservative than ARO.
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- \( w \rho \): makes worst-case estimation for the first part only
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- \( w\rho \): makes worst-case estimation for the first part only
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Time consistency

If $\rho_t|\mathcal{F}_{t-1}$ associated with the any probability distribution $P_t \in \mathcal{P}_t$ is monotone, $t = 1, 2, \cdots, T$, then the corresponding multi-period worst-case risk measure $\{w\rho_{t,T}\}_{t=0}^{T-1}$ is time consistent.
Multi-period worst-case risk measure (Cont’d)

Time consistency

If $\rho_{t|F_{t-1}}$ associated with the any probability distribution $P_t \in \mathcal{P}_t$ is monotone, $t = 1, 2, \cdots, T$, then the corresponding multi-period worst-case risk measure $\{w\rho_{t,T}\}_{t=0}^{T-1}$ is time consistent.

Coherency

If $\rho_{t|F_{t-1}}$ associated with any probability distribution $P_t \in \mathcal{P}_t$ is coherent, the corresponding multi-period worst-case risk measure is dynamic coherent.
Regime-dependent risk measure

Regime switching

Regime-dependent robust risk measures

Introduction Multi-period worst-case risk measure Risk measures

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Regime-dependent robust risk measures

Regime-dependent risk measure

Regime switching

- Regime switching can reflect dynamic correlations of return rates in different economic cycles.
Regime-dependent risk measure

**Regime switching**

- Regime switching can reflect dynamic correlations of return rates in different economic cycles.
- The regime process is \( s_1, \ldots, s_T \).
Regime-dependent risk measure

Regime switching

- Regime switching can reflect dynamic correlations of return rates in different economic cycles.
- The regime process is $s_1, \cdots, s_T$.
- Possible regimes are $s^1, s^2, \cdots, s^J$. 

Stationary Markovian chain with the following transition probability matrix:

$$Q = \begin{pmatrix}
Q_{s_i s_i} & Q_{s_i s_{i+1}} & \cdots & Q_{s_i s_J} \\
Q_{s_{i+1} s_i} & Q_{s_{i+1} s_{i+1}} & \cdots & Q_{s_{i+1} s_J} \\
\vdots & \vdots & \ddots & \vdots \\
Q_{s_J s_i} & Q_{s_J s_{i+1}} & \cdots & Q_{s_J s_J}
\end{pmatrix}.$$
Regime-dependent risk measure

Regime switching

- Regime switching can reflect dynamic correlations of return rates in different economic cycles.
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- Stationary Markovian chain with the following transition probability matrix:

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\vdots & \vdots & \ddots & \vdots \\
Q_{s^J s^1} & Q_{s^J s^2} & \cdots & Q_{s^J s^J}
\end{pmatrix}. $$
Product space
Regime-dependent robust risk measures

Regime-dependent risk measure (Cont’d)

Product space

- Regime process belongs to $(S, S, Q)$, and the corresponding filtration it generates is $S_0 \subseteq S_1 \subseteq \cdots \subseteq S_T$. 
Product space

- Regime process belongs to \((S, S, Q)\), and the corresponding filtration it generates is \(S_0 \subseteq S_1 \subseteq \cdots \subseteq S_T\).
- Consider \(\{x_t, \ t = 0, 1, \cdots, T\}\) on the product space \((\Omega \times S, \mathcal{F} \times S, P \times Q)\).
Product space

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- Consider \(\{x_t, \ t = 0, 1, \cdots, T\}\) on the product space \((\Omega \times S, \mathcal{F} \times S, P \times Q)\).

- At each period \(t, \ t = 0, 1, \cdots, T\), \(x_t\) is adapted to the filtration \(\mathcal{F}_t \times S_t\).
Product space

- Regime process belongs to \((S, S, Q)\), and the corresponding filtration it generates is \(S_0 \subseteq S_1 \subseteq \cdots \subseteq S_T\).
- Consider \(\{x_t, \ t = 0, 1, \cdots, T\}\) on the product space \((\Omega \times S, \mathcal{F} \times S, P \times Q)\).
- At each period \(t, t = 0, 1, \cdots, T\), \(x_t\) is adapted to the filtration \(\mathcal{F}_t \times S_t\).
- From the stationary assumption for \(s_t\), we know that \(Q|S_T \equiv Q|S_t\).
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- Consider \(\{x_t, \ t = 0, 1, \cdots, T\}\) on the product space \((\Omega \times S, F \times S, P \times Q)\).
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Regime-dependent robust risk measures

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- At each period \(t, t = 0, 1, \cdots, T\), \(x_t\) is adapted to the filtration \(\mathcal{F}_t \times S_t\).
- From the stationary assumption for \(s_t\), we know that \(Q|S_\tau \equiv Q|S_t\).

\[\Rightarrow x_t \in L_p(\Omega \times S, \mathcal{F}_t \times S_t, P_t \times Q), \ p \geq 2.\]
To distinguish the influence of $\mathcal{F}_t$ and that of $S_t$. 
To distinguish the influence of \( F_t \) and that of \( S_t \).

**Conditional risk mapping**
To distinguish the influence of $\mathcal{F}_t$ and that of $S_t$.

**Conditional risk mapping**

$$\rho_{t-1,t}(\cdot) : L_p(\Omega \times S, \mathcal{F}_t \times S_t, P_t \times Q) \rightarrow L_p(\Omega \times S, \mathcal{F}_{t-1} \times S_{t-1}, P_{t-1} \times Q)$$

We separate $\rho_{t-1,t}(\cdot)$ into **two levels:**
Regime-dependent risk measure (Cont’d)

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- The conditional risk mapping under given regime $s_t$,

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- The regime-dependent risks are combined by $g_t(\cdot)$:
  $$L_p(\Omega \times S, \mathcal{F}_{t-1} \times S_t, P_{t-1} \times Q) \rightarrow L_p(\Omega \times S, \mathcal{F}_{t-1} \times S_{t-1}, P_{t-1} \times Q)$$
Distributionally robust counterpart
Distributionally robust counterpart

- The uncertainty set $\mathcal{P}_t(s_t)$ at period $t$ is associated with the regime $s_t \in S_t$. 
Distributionally robust counterpart

- The uncertainty set $\mathcal{P}_t(s_t)$ at period $t$ is associated with the regime $s_t \in S_t$.
- With respect to the regime based uncertainty set, the worst-case estimation of the one-period risk at period $t$ is
  \[ w\rho_{s_t}(x_t) = \sup_{P_t \in \mathcal{P}_t(s_t)} \rho_t | F_{t-1} (x_t), \]
Distributionally robust counterpart

- The uncertainty set $\mathcal{P}_t(s_t)$ at period $t$ is associated with the regime $s_t \in S_t$.

- With respect to the regime based uncertainty set, the worst-case estimation of the one-period risk at period $t$ is

$$w\rho_{s_t}(x_t) = \sup_{P_t \in \mathcal{P}_t(s_t)} \rho_t|\mathcal{F}_{t-1}(x_t),$$

Multi-period worst-regime risk measure: find the worst-regime, and the multi-period robust risk measures are formulated in a SEC way.
Regime-dependent risk measure (Cont’d)

Multi-period worst-regime risk measure

For $t = 0, 1, \cdots, T - 1$ and $x_{t+1,T} \in \mathcal{L}_{t+1,T}$,

$$wr\rho_{t,T}(x_{t+1,T}; s_t) = \sum_{i=t+1}^{T} \mathbb{E} \left[ \sup_{s_i \in S_i} \sup_{P_i \in \mathcal{P}_i(s_i)} \rho_{i|F_{i-1}}(x_i) \middle| F_t \times S_t \right]$$

is called the conditional worst-regime risk mapping. And the sequence of the conditional worst-regime risk mappings $\{wr\rho_{t,T}\}_{t=0}^{T-1}$ is called the multi-period worst-regime risk measure.
Regime-dependent risk measure (Cont’d)

Multi-period worst-regime risk measure

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$wr_\rho$ cares about the worst regime and ignores other regimes, a very conservative risk evaluation.
Multi-period worst-regime risk measure

For $t = 0, 1, \ldots, T - 1$ and $x_{t+1,T} \in \mathcal{L}_{t+1,T},$

$$w_{t,T}(x_{t+1,T}; s_t) = \sum_{i=t+1}^{T} \mathbb{E} \left[ \sup_{s_i \in S_i} \sup_{P_i \in \mathcal{P}_i(s_i)} \rho_i|\mathcal{F}_{i-1}(x_i)\right]_{\mathcal{F}_t \times \mathcal{S}_t}$$

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⇒

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Multi-period worst-regime risk measure

For $t = 0, 1, \cdots, T - 1$ and $x_{t+1,T} \in \mathcal{L}_{t+1,T}$,

$$wr_\rho_{t,T}(x_{t+1,T}; s_t) = \sum_{i=t+1}^{T} \mathbb{E} \left[ \sup_{s_i \in S_i} \sup_{P_i \in \mathcal{P}_i(s_i)} \rho_{i|F_{i-1}}(x_i) \bigg| F_t \times S_t \right]$$

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$wr_\rho$ cares about the worst regime and ignores other regimes, a very conservative risk evaluation.

$\Rightarrow$ Weight all sub worst-case risk measures under different regimes.
Multi-period mixed worst-case risk measure

For \( t = 0, 1, \ldots, T - 1 \) and \( x_{t+1,T} \in \mathcal{L}_{t+1,T} \),

\[
mw\rho_{t,T}(x_{t+1,T}; s_t) = \sum_{i=t+1}^{T} E \left[ E \left[ \sup_{P_i \in \mathcal{P}_i(s_i)} \rho_i | \mathcal{F}_{i-1}(x_i) | S_{i-1} \left| \mathcal{F}_t \times S_t \right. \right] \right]
\]

is called the conditional mixed worst-case risk mapping. And the sequence of the conditional mixed worst-case risk mappings \( \{mw\rho_{t,T}\}_{t=0}^{T-1} \) is called the multi-period mixed worst-case risk measure.
Multi-period mixed worst-case risk measure

For \( t = 0, 1, \cdots, T - 1 \) and \( x_{t+1,T} \in \mathcal{L}_{t+1,T} \),

\[
mw\rho_{t,T}(x_{t+1,T}; s_t) = \sum_{i=t+1}^{T} \mathbb{E} \left[ \mathbb{E} \left[ \sup_{P_i \in \mathcal{P}_i(s_i)} \rho_i|\mathcal{F}_{i-1}(x_i)|S_{i-1} \right] | F_t \times S_t \right]
\]

is called the conditional mixed worst-case risk mapping. And the sequence of the conditional mixed worst-case risk mappings \( \{mw\rho_{t,T}\}_{t=0}^{T-1} \) is called the multi-period mixed worst-case risk measure.

\( mw\rho \) takes the information under all regimes into consideration.
Dynamic formulations
Dynamic formulations

\[ \text{wr} \rho_{t-1,T}(x_{t,T}; s_{t-1}) = \left( \sup_{s_t \in S_t} \left( \sup_{P_t \in \mathcal{P}_t(s_t)} \rho_t|\mathcal{F}_{t-1}(x_t) \right) \right) + \mathbb{E} \left[ \text{wr} \rho_{t,T}(x_{t+1,T}; s_t)|\mathcal{F}_{t-1} \times S_{t-1} \right], t = 1, 2, \cdots, T. \]
Regime-dependent risk measure (Cont’d)

Dynamic formulations

\[ wr\rho_{t-1,T}(x_t,T; s_{t-1}) = \left( \sup_{s_t \in S_t} \left( \sup_{P_t \in \mathcal{P}_t(s_t)} \rho_t|\mathcal{F}_{t-1}(x_t) \right) \right) + \mathbb{E}\left[ wr\rho_{t,T}(x_{t+1},T; s_t)|\mathcal{F}_{t-1} \times S_{t-1} \right], \ t = 1, 2, \cdots, T. \]

\[ mw\rho_{t-1,T}(x_t,T; s_{t-1}) = \left( \mathbb{E}\left[ \sup_{P_t \in \mathcal{P}_t(s_t)} \rho_t|\mathcal{F}_{t-1}(x_t)|S_{t-1} \right] \right) + \mathbb{E}\left[ mw\rho_{t,T}(x_{t+1},T; s_t)|\mathcal{F}_{t-1} \times S_{t-1} \right], \ t = 1, 2, \cdots, T. \]
Regime-dependent robust risk measures

Introduction
Multi-period worst-case risk measure
Risk measures
Applications
Empirical illustrations
Conclusions

Regime-dependent risk measure (Cont’d)

Dynamic formulations

\[
wr\rho_{t-1,T}(x_t,T; s_{t-1}) = \left( \sup_{s_t \in S_t} \left( \sup_{P_t \in P_t(s_t)} \rho_{t|F_{t-1}}(x_t) \right) \right) \\
+ \mathbb{E} \left[ wr\rho_{t,T}(x_{t+1},T; s_t)|F_{t-1} \times S_{t-1} \right], t = 1, 2, \cdots, T.
\]

\[
mw\rho_{t-1,T}(x_t,T; s_{t-1}) = \left( \mathbb{E} \left[ \sup_{P_t \in P_t(s_t)} \rho_{t|F_{t-1}}(x_t)|S_{t-1} \right] \right) \\
+ \mathbb{E} \left[ mw\rho_{t,T}(x_{t+1},T; s_t)|F_{t-1} \times S_{t-1} \right], t = 1, 2, \cdots, T.
\]

⇒
Regime-dependent robust risk measure (Cont’d)

Dynamic formulations

\[ wr\rho_{t-1,T}(x_t,T; s_{t-1}) = \left( \sup_{s_t \in S_t} \left( \sup_{P_t \in \mathcal{P}_t(s_t)} \rho_t|\mathcal{F}_{t-1}(x_t) \right) \right) \]
\[ + \mathbb{E} [ wr\rho_t,T(x_{t+1},T; s_t)|\mathcal{F}_{t-1} \times S_{t-1}] , t = 1, 2, \cdots , T. \]

\[ mw\rho_{t-1,T}(x_t,T; s_{t-1}) = \left( \mathbb{E} \left[ \sup_{P_t \in \mathcal{P}_t(s_t)} \rho_t|\mathcal{F}_{t-1}(x_t)|S_{t-1} \right] \right) \]
\[ + \mathbb{E} [ mw\rho_t,T(x_{t+1},T; s_t)|\mathcal{F}_{t-1} \times S_{t-1}] , t = 1, 2, \cdots , T. \]

⇒ time consistency of the two multi-period robust risk measures.
Multi-period robust portfolio selection model under wCVaR (Mean-wCVaR model)

Market setting
Multi-period robust portfolio selection model under wCVaR (Mean-wCVaR model)

Market setting

- There are $n$ risky assets in the security market
Multi-period robust portfolio selection model under wCVaR (Mean-wCVaR model)

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- There are \( n \) risky assets in the security market
- \( r_t = [r_t^1, \cdots, r_t^n]^\top \): the random return rates at period \( t \)
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- $r_t = [r^1_t, \cdots, r^n_t]^\top$: the random return rates at period $t$
- $u_{t-1} = [u^1_{t-1}, \cdots, u^n_{t-1}]^\top$: the vector of cash amounts invested in the risky assets at the beginning of period $t$
Multi-period robust portfolio selection model under wCVaR (Mean-wCVaR model)

Market setting

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$$
\mathcal{P}_t = \left\{ P \left| \mathbb{E}_{P_{t-1}}[r_t] = \mu_t, \text{Cov}_{P_{t-1}}[r_t] = \Gamma_t \right. \right\}.
$$
Mean-wCVaR model

We consider a multi-criteria approach with respect to the expected final wealth and wCVaR measure as follows:
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\[
\begin{align*}
\max_{u} & \quad \mathbb{E}[w_T] - \lambda \cdot \sum_{t=1}^{T} \mathbb{E}\left[ \sup_{P_t \in \mathcal{P}_t} \text{CVaR}_{t|F_{t-1}}(-w_t) \right], \\
\text{s.t.} & \quad e^T u_{t-1} = w_{t-1}, \quad t = 1, \cdots, T. \\
& \quad r_t^T u_{t-1} = w_t, \quad t = 1, \cdots, T.
\end{align*}
\]

Here, \(e^T = [1, \cdots, 1]^T\). \(\lambda\) is the risk aversion coefficient.
Mean-wCVaR model

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\max_u \mathbb{E}[w_T] - \lambda \cdot \sum_{t=1}^{T} \mathbb{E}\left[ \sup_{P_t \in \mathcal{P}_t} \text{CVaR}_{t|\mathcal{F}_{t-1}}(-w_t) \right],
\]

s.t. \quad e^T u_{t-1} = w_{t-1}, \quad t = 1, \cdots, T.

\quad r_t^T u_{t-1} = w_t, \quad t = 1, \cdots, T.

Here, \( e = [1, \cdots, 1]^T \). \( \lambda \) is the risk aversion coefficient.
Mean-wCVaR model (Cont’d)

With the following notations:
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\[ a_t = e^\top \Gamma_t^{-1} e, \quad b_t = e^\top \Gamma_t^{-1} \mu_t, \quad c_t = \mu_t^\top \Gamma_t^{-1} \mu_t, \]

\[ \kappa_t = \sqrt{\frac{1 - \epsilon_t}{\epsilon_t}}, \quad t = 1, \cdots, T, \quad z_T = 1, \]

\[ z_{t-1} = (\lambda + z_t)d_t - \lambda \kappa_t \sqrt{\frac{1}{a_t c_t - b_t^2}}(c_t^2 - 2b_t s_t + a_t s_t^2), \quad t = 2, \cdots, T, \]

\[ h_t = \left( \frac{\lambda \kappa_t}{\lambda + z_t} \right)^2 \frac{1}{a_t c_t - b_t^2}, \quad \Delta_t = 4(h_t a_t - 1)(a_t c_t - b_t^2), \]

\[ d_t = \frac{2b(a_t h_t - 1) + \sqrt{\Delta_t}}{2a_t(a_t h_t - 1)}, \quad t = 1, \cdots, T. \]
Mean-wCVaR model (Cont’d)

With the following notations:

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\[ h_t = \left( \frac{\lambda \kappa_t}{\lambda + z_t} \right)^2 \frac{1}{a_tc_t - b_t^2}, \quad \Delta_t = 4(h_ta_t - 1)(a_tc_t - b_t^2), \]
\[ d_t = \frac{2b(a_th_t - 1) + \sqrt{\Delta_t}}{2a_t(a_th_t - 1)}, \quad t = 1, \cdots, T. \]

we can solve the mean-wCVaR problem analytically.
Theorem

Suppose that the wealth \( w_t \) at each period \( t \) is non-negative, and the investor is risk averse such that \( \lambda + z_t \) is always non-negative. Then, if \( a_t h_t - 1 \geq 0 \) for all \( t = 1, \cdots, T \), the optimal investment policy for problem (4)-(6) is

\[
 u_{t-1} = \left( \Gamma_t^{-1} e \Gamma_t^{-1} \mu_t \right) \frac{1}{a_t c_t - b_t^2} \begin{pmatrix} c_t & -b_t \\ -b_t & a_t \end{pmatrix} \begin{pmatrix} 1 \\ d_t \end{pmatrix} w_{t-1}, \ t = 1, \cdots, T. 
\]

If \( a_t h_t - 1 < 0 \) for some \( t, 1 \leq t \leq T \), the optimal portfolio at period \( t - 1 \) trends to infinity, and the problem (4)-(6) is unbounded.
Mean-mwCVaR and mean-wrCVaR models

The mean-mwCVaR model with transaction costs and market restriction constraints.
Mean-mwCVaR and mean-wrCVaR models

The mean-mwCVaR model with transaction costs and market restriction constraints.

$$\max_u \left\{ \mathbb{E}[w_T; s_0] - \lambda \cdot mwCVaR_{0,T}(-w_{1,T}; s_0) \right\},$$

s.t.  
$$w_0 = u_0^\top e + \alpha^\top (u_0)^+ + \beta^\top (u_0)^-, \quad w_t = u_t^\top e + \alpha^\top (u_t - u_{t-1})^+ + \beta^\top (u_t - u_{t-1})^-, \quad t = 1, \cdots, T - 1,$$

$$w_{t+1} = u_t^\top r_{t+1}, \quad t = 0, \cdots, T - 1,$$

$$\underline{u} \leq u_t \leq \bar{u}, \quad t = 0, \cdots, T - 1,$$
The mean-wrCVaR model with transaction costs and market restriction constraints.
Mean-mwCVaR and mean-wrCVaR models

The **mean-wrCVaR model** with transaction costs and market restriction constraints.

\[
\begin{align*}
\max_u \{ & \mathbb{E}[w_T; s_0] - \lambda \cdot \text{wrCVaR}_{0,T}(-w_{1,T}; s_0), \\
\text{s.t.} \quad & w_0 = u_0^T e + \alpha^T (u_0)^+ + \beta^T (u_0)^-, \\
& w_t = u_t^T e + \alpha^T (u_t - u_{t-1})^+ + \beta^T (u_t - u_{t-1})^-, t = 1, \ldots, T - 1, \\
& w_{t+1} = u_t^T r_{t+1}, t = 0, \ldots, T - 1, \\
& \underline{u} \leq u_t \leq \bar{u}, t = 0, \ldots, T - 1,
\end{align*}
\]
We adopt a scenario tree to transform the mean-mwCVaR and mean-wrCVaR models.
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Some notations:

- $K^+$: the set of all nodes at periods 1, 2, ... , $T$;
Mean-mwCVaR and mean-wrCVaR models (Cont’d)

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- $K^+$: the set of all nodes at periods $1, 2, \cdots, T$;
- $N(K^+)$: the number of nodes in $K^+$;
Mean-mwCVaR and mean-wrCVaR models (Cont’d)

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- $K^+$: the set of all nodes at periods $1, 2, \cdots, T$;
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- $K^-$: the set of all nodes at periods $0, 1, \cdots, T - 1$;
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- $N(K^-)$: the number of nodes in $K^-$;
- $t(k)$: the number of period of node $k$;
Mean-mwCVaR and mean-wrCVaR models (Cont’d)

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- $t(k)$: the number of period of node $k$;
- $s(k)$: the regime of node $k$;
Some notations:

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- $N(K^-)$: the number of nodes in $K^-$;
- $t(k)$: the number of period of node $k$;
- $s(k)$: the regime of node $k$;
- $Q(k; s_0)$: node $k$’s appearing probability in the tree.
For a node $k \in K^+$, the unique predecessor is denoted as $k^-$;
Mean-mwCVaR and mean-wrCVaR models (Cont’d)

- For a node \( k \in K^+ \), the unique predecessor is denoted as \( k^- \);
- \( \mu(k) \): the estimated expectation value of \( r_t \) at node \( k \);
Mean-mwCVaR and mean-wrCVaR models (Cont’d)

- For a node $k \in K^+$, the unique predecessor is denoted as $k^-$;
- $\mu(k)$: the estimated expectation value of $r_t$ at node $k$;
- $\Gamma(k)$: the estimation value of the conditional covariance matrix;
Mean-mwCVaR and mean-wrCVaR models (Cont’d)

- For a node $k \in K^+$, the unique predecessor is denoted as $k^-$;
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- The uncertainty set with respect to the regime $s(k)$
For a node $k \in K^+$, the unique predecessor is denoted as $k^-$;

- $\mu(k)$: the estimated expectation value of $r_t$ at node $k$;
- $\Gamma(k)$: the estimation value of the conditional covariance matrix;

The uncertainty set with respect to the regime $s(k)$

$$\mathcal{P}(k) = \left\{ P \left| \mathbb{E}_{P_{t-1}} [r_t | \mathcal{F}_{t-1}, s_t = s(k)] = \mu(k), \right. \right.$$ 

$$\left. \Gamma_{P_{t-1}} [r_t | \mathcal{F}_{t-1}, s_t = s(k)] = \Gamma(k) \right\}.$$
Under the scenario tree setting, the mean-mwCVaR model is equivalent to the following cone programming problem:
Under the scenario tree setting, the mean-mwCVaR model is equivalent to the following cone programming problem:

**Object:**
Mean-mwCVaR model (Cont’d)

Under the scenario tree setting, the mean-mwCVaR model is equivalent to the following cone programming problem:

Object:

$$
\max_{u, y, z, g, u^+, u^-} \left\{ (1 + \lambda)w_0 + \sum_{k \in K^+} (1 + (T - t(k^-) - 1)\lambda)Q(k; s_0)(\mu(k) - e)^T u(k^-) \\
- \lambda \sum_{k \in K^+} Q(k; s_0)y(k) - (1 + T\lambda)(\alpha^T u^+(0) + \beta^T u^-(0)) \\
- \sum_{k \in K^- \setminus \{0\}} (1 + (T - t(k))\lambda)[\alpha^T u^+(k) + \beta^T u^-(k)] \right\}
$$
Mean-mwCVaR model (Cont’d)

Constraints:
Mean-mwCVaR model (Cont’d)

Constraints:

\[
\begin{align*}
\Gamma^{1/2}(k)u(k^-) &= z(k), \quad k \in K^+,
\mu(k) - e)^\top u(k^-) + y(k) &= \kappa(k)g(k), \quad k \in K^+,
\|z(k)\|_2 &\leq g(k), \quad k \in K^+,
u(0) &= u^+(0) - u^-(0),
w_0 &= u(0)^\top e + \alpha^\top u^+(0) + \beta^\top u^-(0),
u(k) - u(k^-) &= u^+(k) - u^-(k), \quad k \in K^- \setminus \{0\},
u(k^-)^\top \mu(k) &= u(k)^\top e + \alpha^\top u^+(k) + \beta^\top u^-(k), \quad k \in K^- \setminus \{0\},
u^+(k), u^-(k) &\geq 0, \quad k \in K^-,\underline{u} &\leq u(k) \leq \bar{u}, \quad k \in K^-,
\end{align*}
\]
Mean-mwCVaR model (Cont’d)

Constraints:

\[ \Gamma^{1/2}(k)u(k^-) = z(k), \ k \in K^+, \]
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The above SOCP has \((n + 2)N(K^+) + 3nN(K^-)\) variables, \((n + 1)N(K^+) + (n + 1)N(K^-)\) linear constraints and \(N(K^+)\) standard second order cone constraints.
Under the scenario tree setting, the mean-wrCVaR model is equivalent to the following cone programming problem:
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**Object:**
Mean-wrCVaR model (Cont’d)

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$$- \lambda \sum_{k \in K^-} Q(k; s_0)y(k) - (1 + T\lambda)(\alpha^T u^+(0) + \beta^T u^-(0))$$  

$$+ \sum_{k \in K^- \setminus \{0\}} (1 + (T - t(k))\lambda)[\alpha^T u^+(k) + \beta^T u^-(k)] \right\}$$
Mean-wrCVaR model (Cont’d)

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We compare the following three dynamic portfolio selection models:

- wCVaR: mean-wCVaR model
- MV: dynamic MV model in Li et al. (2000)
- LPM2: multistage portfolio selection model with robust second-order lower partial moment (LPM2) as the risk measure in Chen et al. (2011)

We simulated the models for 100 times. Use mean and variance in Example 2 of Li et al. (2000). Generate return rate samples by Gaussian Distribution. T = 4.
Simulation results

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Characteristics of the terminal wealths among 100 groups of samples
Characteristics of the terminal wealths among 100 groups of samples

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<th></th>
<th>mean</th>
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<th>MV</th>
<th>LPM2</th>
<th>variance</th>
<th>wCVaR</th>
<th>MV</th>
<th>LPM2</th>
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<tr>
<td></td>
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<td>MV</td>
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<td></td>
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<td>1.8296</td>
<td>1.1875</td>
<td>0.2162</td>
<td>504.9351</td>
<td>0.1466</td>
<td></td>
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</tr>
</tbody>
</table>
Simulation results (Cont’d)

- MV model gains high wealth under best cases, and suffers extreme large loss under worst cases.
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- MV model gains high wealth under best cases, and suffers extreme large loss under worst cases.
- When the actual distribution has bias from Gaussian (extreme cases), MV model performs badly.
- Robust technique can efficiently reduce the expected wealth loss and investment risk under extreme cases.
- wCVaR model is not that extremely conservative as the LPM2 model, and it makes a good balance between providing a high terminal wealth and controlling the extreme risk.
Empirical results

Market setting (Dow Jones, S & P500)
Empirical results

**Market setting** (Dow Jones, S & P500)

- 10 stocks from different industries in American stock markets
Empirical results

Market setting (Dow Jones, S & P500)

- 10 stocks from different industries in American stock markets
- We use adjusted daily close-prices of these stocks on every Monday to compute their weekly logarithmic return rates from February 14, 1977 to January 30, 2012
Empirical results

Market setting (Dow Jones, S & P500)

- 10 stocks from different industries in American stock markets
- We use adjusted daily close-prices of these stocks on every Monday to compute their weekly logarithmic return rates from February 14, 1977 to January 30, 2012
- We divide the market into three regimes: the bull regime; the consolidation regime and the bear regime
Empirical results (Cont’d)

Determining regime (NYSF, AMEX, NASDAQ)

Use MKT - RF (Fama and French, 1993) to determine regime

Effective time window with 28 weeks, centered on the examining week

Add all MKT - RF in the effective time window and compare with pre-set benchmark

Sum larger than 1.0 ⇒ bull regime

Sum smaller than -1.0 ⇒ bear regime

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Empirical results (Cont’d)

Estimating regime transition probability
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Estimating regime transition probability

Counting the relevant historical transition times
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\[ Q = \begin{bmatrix}
0.9475 & 0.0336 & 0.0189 \\
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Empirical results (Cont’d)

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- Stable to stay in the bull or bear regime
Empirical results (Cont’d)

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- Stable to stay in the bull or bear regime
- High possibility to switch from the consolidation regime into the bull or bear regime
Empirical results (Cont’d)

Expected return rates (%) under different regimes

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<td>$\mu(s^1)$</td>
<td>0.2486</td>
<td>0.1845</td>
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Both first and second order moments have significant difference among different regimes. The estimated covariance matrices have the same feature.
### Empirical results (Cont’d)

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Find the optimal portfolios of mean-wCVaR mean-wrCVaR, mean-mwCVaR models by solving the SOCPs
Empirical results (Cont’d)

Find the optimal portfolios of mean-wCVaR mean-wrCVaR, mean-mwCVaR models by solving the SOCPs

Root optimal portfolios

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<td>$u_{wCVaR}^*(s_0)$</td>
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<td>0.3000</td>
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<td>0.3000</td>
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<tr>
<td>$u^*_{wCVaR}(s_0)$</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.1005</td>
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</tr>
<tr>
<td>$u^*_{wrCVaR}(s_0)$</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.1367</td>
<td>0.2633</td>
<td></td>
</tr>
<tr>
<td>$u^*_{mwCVaR}(s_0 = s^1)$</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.1385</td>
<td>0.0000</td>
<td>0.2615</td>
<td>0.0000</td>
</tr>
<tr>
<td>$u^*_{mwCVaR}(s_0 = s^2)$</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.0550</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0450</td>
</tr>
<tr>
<td>$u^*_{mwCVaR}(s_0 = s^3)$</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.1492</td>
<td>0.2508</td>
</tr>
</tbody>
</table>

$\epsilon_t(s_t) = 0.05$, $\lambda = 20$, $\underline{u} = 0$, $\overline{u} = 0.3$. 
Both the optimal portfolios of mean-wVaR model and mean-wrVaR model do not rely on the current regime.
Empirical results (Cont’d)

- Both the optimal portfolios of mean-wVaR model and mean-wrVaR model do not rely on the current regime.

- The mean-mwVaR model provides us with three optimal portfolios under three different regimes.
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- That is because the estimation of mwVaR relies on the regime appearing probability in the future.

- The strategy derived under regime-dependent robust models reveals more information about market regimes than the traditional worst-case risk measures.
Out-of-sample test

In-sample period
Out-of-sample test

In-sample period

Out-of-sample test

In-sample period


Out-of-sample period
Out-of-sample test

In-sample period


Out-of-sample period

Out-of-sample test

In-sample period


Out-of-sample period


Rolling forward weekly
Out-of-sample test

In-sample period


Out-of-sample period


Rolling forward weekly

- 100 out-of-sample weekly return rates
Out-of-sample performances

We carry out the out-of-sample test by rolling forward for 100 weeks, this provides us three out-of-sample accumulated wealth series.
We carry out the out-of-sample test by rolling forward for 100 weeks, this provides us three out-of-sample accumulated wealth series.
### Out-of-sample performances (Cont’d)

#### Statistics of out-of-sample performances

<table>
<thead>
<tr>
<th>model</th>
<th>mean-wCVaR</th>
<th>mean-wrCVaR</th>
<th>mean-mwCVaR</th>
</tr>
</thead>
<tbody>
<tr>
<td>maximum (%)</td>
<td>1.1020</td>
<td>1.0683</td>
<td>1.2713</td>
</tr>
<tr>
<td>minimum (%)</td>
<td>-1.4588</td>
<td>-1.4586</td>
<td>-1.2030</td>
</tr>
<tr>
<td>mean (%)</td>
<td>0.1229</td>
<td>0.1234</td>
<td>0.1627</td>
</tr>
<tr>
<td>variance (×1.0e-4)</td>
<td>0.2639</td>
<td>0.2688</td>
<td>0.2957</td>
</tr>
<tr>
<td>skewness</td>
<td>-0.4449</td>
<td>-0.4343</td>
<td>-0.1873</td>
</tr>
</tbody>
</table>

Mean-wCVaR and mean-wrCVaR models have similar performance. Mean-mwCVaR model provides much higher return rate than the other two in terms of the maximum and mean.
### Out-of-sample performances (Cont’d)

#### Statistics of out-of-sample performances

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<tr>
<th>model</th>
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- Mean-wCVaR and mean-wrCVaR models have similar performance
Out-of-sample performances (Cont’d)

Statistics of out-of-sample performances

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<th>model</th>
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- Mean-wCVaR and mean-wrCVaR models have similar performance.
- Mean-mwCVaR model provides much higher return rate than the other two in terms of the maximum and mean.
<table>
<thead>
<tr>
<th>model</th>
<th>regime</th>
<th>bull</th>
<th>consolidation</th>
<th>bear</th>
</tr>
</thead>
<tbody>
<tr>
<td>mean-wCVaR</td>
<td>mean (%)</td>
<td>0.1421</td>
<td>0.2729</td>
<td>0.0339</td>
</tr>
<tr>
<td></td>
<td>variance (×1.0e-4)</td>
<td>0.2455</td>
<td>0.3133</td>
<td>0.3129</td>
</tr>
<tr>
<td>mean-wrCVaR</td>
<td>mean (%)</td>
<td>0.1370</td>
<td>0.2401</td>
<td>0.0579</td>
</tr>
<tr>
<td></td>
<td>variance (×1.0e-4)</td>
<td>0.2542</td>
<td>0.3230</td>
<td>0.3129</td>
</tr>
<tr>
<td>mean-mwCVaR</td>
<td>mean (%)</td>
<td>0.1938</td>
<td>0.2588</td>
<td>0.0535</td>
</tr>
<tr>
<td></td>
<td>variance (×1.0e-4)</td>
<td>0.2902</td>
<td>0.3421</td>
<td>0.3087</td>
</tr>
</tbody>
</table>

Under consolidation market: All three are similar
Under bear market: mean-wrCVaR is best
Under bull market: mean-mwCVaR is best
Out-of-sample performances under different regimes

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<th>bear</th>
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</thead>
<tbody>
<tr>
<td>mean-wCVaR</td>
<td>weight (weeks)</td>
<td>69</td>
<td>6</td>
<td>25</td>
</tr>
<tr>
<td></td>
<td>mean (%)</td>
<td>0.1421</td>
<td>0.2729</td>
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- Under consolidation market: All three are similar
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- Under consolidation market: All three are similar
- Under bear market: **mean-wrCVaR is best**
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- Under consolidation market: All three are similar
- Under bear market: mean-wrCVaR is best
- Under bull market: mean-mwCVaR is best
Different sizes of stock pools:

1. 10 stocks from Dow Jones IA, S & P 500
2. 50 stocks from S & P 500
3. 100 stocks from S & P 500

Adjusted daily close-prices to compute their daily logarithmic return rates from March 20, 2011 to March 3, 2015.
Different sizes of stock pools:

- 10 stocks from Dow Jones IA, S & P 500

Adjusted daily close-prices to compute their daily logarithmic return rates from March 20, 2011 to March 3, 2015.
Different sizes of stock pools:

- 10 stocks from Dow Jones IA, S & P 500
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Adj usted daily close-prices to compute their daily logarithmic return rates from March 20, 2011 to March 3, 2015.
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Different sizes of stock pools:

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- Adjusted daily close-prices to compute their daily logarithmic return rates from March 20, 2011 to March 3, 2015
Separate the historical daily data into:
Different sizes of stock pools

Separate the historical daily data into:

- The in-sample period: March 20, 2011 to October 7, 2014
Separate the historical daily data into:

- The in-sample period: March 20, 2011 to October 7, 2014
Different sizes of stock pools

Separate the historical daily data into:

- The in-sample period: March 20, 2011 to October 7, 2014

Divide the market into three regimes:
Separate the historical daily data into:

- The in-sample period: March 20, 2011 to October 7, 2014

Divide the market into three regimes:

- Using the effective time window method stated above
Different sizes of stock pools

Separate the historical daily data into:

- The in-sample period: March 20, 2011 to October 7, 2014

Divide the market into three regimes:

- Using the effective time window method stated above
- In the out-of-sample period:
Different sizes of stock pools

Separate the historical daily data into:

- The in-sample period: March 20, 2011 to October 7, 2014

Divide the market into three regimes:

- Using the effective time window method stated above
- In the out-of-sample period:
  - Bull regime: 68 days
Different sizes of stock pools

Separate the historical daily data into:

- The in-sample period: March 20, 2011 to October 7, 2014

Divide the market into three regimes:

- Using the effective time window method stated above
- In the out-of-sample period:
  - Bull regime: 68 days
  - Consolidation regime: 15 days
Separate the historical daily data into:

- The in-sample period: March 20, 2011 to October 7, 2014

Divide the market into three regimes:

- Using the effective time window method stated above
- In the out-of-sample period:
  - Bull regime: 68 days
  - Consolidation regime: 15 days
  - Bear regime: 17 days
### Different sizes of stock pools

Statistics of out-of-sample return series got under three models with different stocks pools

<table>
<thead>
<tr>
<th>mean-wCVaR</th>
<th>10 stocks</th>
<th>50 stocks</th>
<th>100 stocks</th>
</tr>
</thead>
<tbody>
<tr>
<td>total</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>mean (%)</td>
<td>0.0331</td>
<td>0.0473</td>
<td>0.0771</td>
</tr>
<tr>
<td>variance (×10e-4)</td>
<td>0.608</td>
<td>0.639</td>
<td>0.728</td>
</tr>
<tr>
<td>bull</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>mean (%)</td>
<td>0.001</td>
<td>-0.0483</td>
<td>-0.0494</td>
</tr>
<tr>
<td>variance (×10e-4)</td>
<td>0.5415</td>
<td>0.7933</td>
<td>1.2447</td>
</tr>
<tr>
<td>consolidation</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>mean (%)</td>
<td>0.5026</td>
<td>0.528</td>
<td>0.5006</td>
</tr>
<tr>
<td>variance (×10e-4)</td>
<td>0.4668</td>
<td>0.3368</td>
<td>0.4225</td>
</tr>
<tr>
<td>bear</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>mean (%)</td>
<td>-0.2565</td>
<td>0.0006</td>
<td>0.1164</td>
</tr>
<tr>
<td>variance (×10e-4)</td>
<td>0.8118</td>
<td>0.7361</td>
<td>1.0421</td>
</tr>
</tbody>
</table>
### Different sizes of stock pools

Statistics of out-of-sample return series got under three models with different stocks pools

<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td><strong>total</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>mean (%)</td>
<td>0.0324</td>
<td>0.0465</td>
<td>0.0613</td>
</tr>
<tr>
<td>variance (×10e-4)</td>
<td>0.612</td>
<td>0.745</td>
<td>1.109</td>
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<td><strong>bull</strong></td>
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</tr>
<tr>
<td>mean (%)</td>
<td>0.0001</td>
<td>-0.0321</td>
<td>0.0585</td>
</tr>
<tr>
<td>variance (×10e-4)</td>
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<tr>
<td><strong>consolidation</strong></td>
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</tr>
<tr>
<td>mean (%)</td>
<td>0.5029</td>
<td>0.5256</td>
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</tr>
<tr>
<td>variance (×10e-4)</td>
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<td>0.3517</td>
<td>0.6414</td>
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<tr>
<td><strong>bear</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>mean (%)</td>
<td>-0.2492</td>
<td>-0.0572</td>
<td>-0.2489</td>
</tr>
<tr>
<td>variance (×10e-4)</td>
<td>0.8339</td>
<td>0.5223</td>
<td>0.5315</td>
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### Statistics of out-of-sample return series got under three models with different stocks pools

<table>
<thead>
<tr>
<th></th>
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<th>10 stocks</th>
<th>50 stocks</th>
<th>100 stocks</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>mean (%)</td>
<td>0.0370</td>
<td>0.0817</td>
<td>0.0855</td>
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<tr>
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<td>0.621</td>
<td>0.739</td>
<td>1.072</td>
</tr>
<tr>
<td>total</td>
<td>mean (%)</td>
<td>0.0078</td>
<td>0.006</td>
<td>-0.0143</td>
</tr>
<tr>
<td></td>
<td>variance (×10e-4)</td>
<td>0.5522</td>
<td>0.7751</td>
<td>1.1805</td>
</tr>
<tr>
<td>bull</td>
<td>mean (%)</td>
<td>0.4995</td>
<td>0.535</td>
<td>0.5345</td>
</tr>
<tr>
<td></td>
<td>variance (×10e-4)</td>
<td>0.4839</td>
<td>0.4224</td>
<td>0.4313</td>
</tr>
<tr>
<td>consolidation</td>
<td>mean (%)</td>
<td>-0.2545</td>
<td>-0.0154</td>
<td>0.0885</td>
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<td>0.7317</td>
<td>1.0806</td>
</tr>
<tr>
<td>bear</td>
<td>mean (%)</td>
<td>0.4995</td>
<td>0.535</td>
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- The solution times for the encountered SOCP problems with 10 stocks are between 0.42 seconds and 0.55 seconds;
Different sizes of stock pools

- The solution times for the encountered SOCP problems with 10 stocks are between 0.42 seconds and 0.55 seconds;
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Different sizes of stock pools

- The solution times for the encountered SOCP problems with 10 stocks are between 0.42 seconds and 0.55 seconds;

- The solution times for the encountered SOCP problems with 50 stocks are between 0.45 seconds and 1.59 seconds;

- The solution times for the encountered SOCP problems with 100 stocks are between 0.55 seconds and 7.60 seconds.
The out-of-sample accumulative wealth series got under the mean-wCVaR model
The out-of-sample accumulative wealth series got under the mean-wrCVaR model
The out-of-sample accumulative wealth series got under the mean-mwCVaR model.
Different sizes of stock pools

- The mean-mwCVaR model constantly provides much greater return rate than the other two models, independently of the three stock pools.
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- As the size of the stock pool becomes larger and larger, the out-of-sample return rates got under the three models generally become greater too.
Different sizes of stock pools

When the market is:

Under the bull regime, the portfolio selection models with a smaller stock pool perform better;
Under the consolidation regime, the performance of the portfolio selection models with a smaller stock pool is similar to that of the portfolio selection models with a larger stock pool;
Under the bear regime, the portfolio selection models with a larger stock pool significantly perform better.
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During a medium-term or long-term real investment process:

- When the investor finds that the market is constantly going high, he/she can focus on the best performing stocks and balance his/her investment among them;
- When he/she finds that the market is turning down, the investor should diversify his/her investment in more assets even if the performance of some assets is not so good as the best performing stocks temporarily;
- Enlarging the stock pool and adopting the multi-period robust portfolio selection model can efficiently avoid the large risks which the investor may suffer under bad market regimes.
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Conclusions

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Numerical results demonstrate the efficiency and flexibility of the proposed models.
Thank You Very Much for Your Attention!